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Poisson brackets, strings and membranes

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Abstract. We construct Poisson brackets at the boundaries of open strings and membranes with constant background fields which are compatible with their boundary conditions. The boundary conditions are treated as primary constraints which give infinitely many secondary constraints. We show explicitly that we need only two (the primary one and one of the secondary ones) constraints to determine the Poisson brackets of strings. We apply this to membranes by using canonical transformations.

1 Introduction

Recently non-commutative spacetime attracted much attention from both the theoretical [1–4] and phenomenological [5] points of view. Especially in string theory, there are many studies of non-commutative descriptions of D-branes which are translated into a commutative description by the Seiberg–Witten map [1].

It was pointed out by Connes, Douglas and Schwarz [2] that M-theory with a constant background 3-form tensor field compactified on a torus can be identified with matrix theory compactified on a non-commutative torus. Similarly, string theory with a background NS B-field is equivalent to string theory on a non-commutative space [3].

The non-commutativity comes from the fact that the canonical Poisson bracket does not coincide with a boundary condition [6]. Some authors have made efforts to obtain Poisson brackets which are compatible with boundary conditions of strings [7–12] and membranes [13,14]. Let us call them "boundary Poisson brackets"[1]. For strings, boundary Poisson brackets can be obtained by using the Dirac formalism [7,8,10]. The quantization is defined by replacement of a Dirac bracket with a commutator; $\{\ ,\ \}_{\mathrm{D}} \to -\mathrm{i}[\ ,\ ]$. When a NS B-field is turned on, $\{X^\mu, X^\nu\}_{\mathrm{D}}$ has a non-zero value at the boundaries, and the boundary coordinates become non-commutative at the quantum level.

In M-theory, the fundamental object is called the "M2-brane" which is a 2-dimensionally extended object. This is coupled with a 3-form field. The boundary condition of the membrane is non-linear, and it is difficult to get all of the secondary constraints. Thus we need some new ideas to deal with the system. In [14], a partial gauge fixing condition with which a boundary constraint of a membrane gives a finite number of constraints is introduced. We would like to determine a boundary Poisson bracket in a completely gauge fixed action. In this paper, keeping this in mind, we construct a boundary Poisson bracket of an open string by avoiding using all of the secondary constraints, and apply this to a membrane.

The present paper is organized as follows. In the next section, we show that it is possible to construct a boundary Poisson bracket of an open string from two constraints by demanding that the canonical Poisson bracket is changed only at boundaries of an open string. In Sect. 3, we put to use the previous procedure in a system of a membrane. Section 4 is devoted to a summary.

2 Strings and constant 2-form fields

In general, the canonical Poisson bracket,

$$\begin{aligned} \{X^\mu(\sigma), X^\nu(\sigma')\}_p &= 0, \\ \{X^\mu(\sigma), P_\nu(\sigma')\}_p &= \delta^\mu_\nu \delta(\sigma - \sigma'), \\ \{P_\mu(\sigma), P_\nu(\sigma')\}_p &= 0, \end{aligned} \tag{2.1}$$

in field theory and string theory with boundaries does not coincide with their boundary conditions. In string theory, since open strings have endpoints, we have to impose their boundary conditions. Both Neumann and Dirichlet conditions change the canonical Poisson bracket at the boundaries of open strings. In the presence of a NS B-field, along a D-brane, a boundary condition of an open string becomes a mixed type of Neumann and Dirichlet conditions. Also the mixed type boundary condition does not coincide with the canonical Poisson structure [6]. With respect to mixed type boundary conditions, it is non-trivial work to determine boundary Poisson brackets. The representative method to construct boundary Poisson brackets is the Dirac formalism.

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[1] After the first version of the present paper was finished, we were informed by Bering that the term "boundary Poisson bracket" has already been used in [15]. Their definition of the term is different from ours

In the Dirac formalism, boundary conditions are dealt with as primary constraints. By the definition of a Dirac bracket, a boundary condition and a Dirac bracket are manifestly compatible with each other. In other words, a Dirac bracket between a boundary condition and canonical variables vanish, if the constraints are of second class. This system is one of the rare examples in which primary constraints produce infinitely many secondary constraints.

On this subject there are some papers [7,8,10]. In this system, we would like to see that we are able to determine a boundary Poisson bracket if we demand locality of the boundary Poisson bracket, i.e. the canonical Poisson bracket is changed only at the boundaries of an open string. In the following, we avoid using the Dirac formalism directly.

The gauge fixed action of a bosonic open string with a constant NS B-field background is given by

$$S = -\frac{T_\mathrm{s}}{2}\int \mathrm{d}^2\xi[\partial_\alpha X^\mu \partial^\alpha X_\mu - B_{\mu\nu}\epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu], \quad (2.2)$$

where T_s is the string tension. The canonical momentum is $P^\mu = T_\mathrm{s}(\partial_\tau X^\mu + B^{\mu\nu}\partial_\sigma X_\nu)$ and the action is non-singular. By the variational principle, the equation of motion is

$$\partial_\alpha \partial^\alpha X^\mu = 0, \quad (2.3)$$

and along the D-brane the boundary condition in terms of the canonical momentum is

$$\int_0^\pi \mathrm{d}\sigma \delta(\sigma)\left(\frac{1}{T_\mathrm{s}}B^{\mu\nu}P_\nu(\sigma) + M^{\mu\nu}\partial_\sigma X_\nu(\sigma)\right) = 0. \quad (2.4)$$

with $M = \eta - B^2$ ($\eta^{\mu\nu}$ the target space flat metric tensor) and similarly with $\delta(\sigma-\pi)$. The canonical Hamiltonian is

$$\mathcal{H} = \frac{T_\mathrm{s}}{2}\left[\left(\frac{1}{T_\mathrm{s}}P_\mu - B_{\mu\nu}\partial_\sigma X^\nu\right)^2 + \partial_\sigma X^\mu \partial_\sigma X_\mu\right]. \quad (2.5)$$

In this paper we will mainly consider the boundary condition only at $\sigma = 0$, since the discussion of the condition at $\sigma = 0$ is parallel with that at $\sigma = \pi$. Boundary Poisson brackets must be compatible with the constraint. We denote the boundary Poisson bracket by $\{\ ,\ \}_\mathrm{b}$ which is defined by the condition that the boundary Poisson bracket of the boundary condition with an arbitrary canonical variable vanishes;

$$\left\{f(X,P), \int_0^\pi \mathrm{d}\sigma' \delta(\sigma') \times \left(\frac{1}{T_\mathrm{s}}B^{\nu\rho}P_\rho(\sigma') + M^{\nu\rho}\partial_{\sigma'}X_\rho(\sigma')\right)\right\}_\mathrm{b} = 0, \quad (2.6)$$

where $f(X,P)$ is an arbitrary function on the phase space. From this condition only, we cannot determine the boundary Poisson bracket uniquely since there are only 2 independent equations for 3 unknowns [6]. In order to determine this uniquely, it was considered that we must use the secondary constraints [7,8]. The boundary constraint (2.4) gives infinitely many secondary constraints [8] which are

$$\int_0^\pi \mathrm{d}\sigma \delta(\sigma)\partial_\sigma^{2n}\left(\frac{1}{T_\mathrm{s}}B^{\mu\nu}P_\nu(\sigma) + M^{\mu\nu}\partial_\sigma X_\nu(\sigma)\right) = 0 \quad (n = 1,2,\cdots), \quad (2.7)$$

$$\int_0^\pi \mathrm{d}\sigma \delta(\sigma)\partial_\sigma^{2n+1}P^\mu(\sigma) = 0 \quad (n = 0,1,\cdots). \quad (2.8)$$

They originate from the condition of stationarity of the boundary constraint (2.4). From these we choose, for example, the $n = 0$ case of (2.8). We have another explanation for the necessity of the condition for the case with $B_{\mu\nu} \neq 0$. We need the condition in order for the equation of motion (2.3) to be equivalent to Hamilton's equations:

$$\partial_\tau X^\mu(\sigma) = \frac{1}{T_\mathrm{s}}P^\mu(\sigma) - B^{\mu\nu}\partial_\sigma X_\nu(\sigma), \quad (2.9)$$

$$\partial_\tau P^\mu(\sigma) = B^{\mu\nu}\partial_\sigma P_\nu + T_\mathrm{s}M^{\mu\nu}\partial_\sigma^2 X_\nu. \quad (2.10)$$

At the boundaries, (2.9) can also be rewritten as

$$\partial_\tau X^\mu(\sigma)\big|_{\sigma=0,\pi} = \frac{1}{T_\mathrm{s}}\,(M^{-1})^{\mu\nu}P_\nu(\sigma)\big|_{\sigma=0,\pi}. \quad (2.11)$$

By virtue of the condition (2.8) with $n = 0$, the equation of motion (2.3) is equivalent to the equations (2.11) and (2.10).

Here a question arises. Do we need all of the secondary constraints to construct the boundary Poisson bracket? In [9] only the equation of motion (2.3) and the boundary condition (2.4) are used. We would like to consider this in the following.

Let us add the condition

$$\left\{f(X,P), \int_0^\pi \mathrm{d}\sigma' \delta(\sigma')\partial_{\sigma'}P^\nu(\sigma')\right\}_\mathrm{b} = 0 \quad (2.12)$$

to the previous one (2.6). Since we have 4 equations for 3 unknowns, in spite of the existence of the infinitely many constraints, we have a possibility to be able to determine the boundary Poisson brackets, if we demand their locality. We assume that the bracket is anti-symmetric and bilinear and satisfies the derivation rule which are fundamental properties of the canonical Poisson bracket.

By solving (2.6) and (2.12) with $f = X, P$ we have

$$\{X^\mu(\sigma), X^\nu(\sigma')\}_\mathrm{b} = \frac{1}{T_\mathrm{s}}Q(\sigma,\sigma')(M^{-1}B)^{\mu\nu}, \quad (2.13)$$

$$\{X^\mu(\sigma), P_\nu(\sigma')\}_\mathrm{b} = \delta^\mu_\nu \hat{\delta}(\sigma,\sigma'), \quad (2.14)$$

$$\{P_\mu(\sigma), P_\nu(\sigma')\}_\mathrm{b} = 0, \quad (2.15)$$

where

$$\partial_\sigma \hat{\delta}(\sigma,\sigma')\Big|_{\sigma=0,\pi} = 0, \quad \partial_\sigma Q(\sigma,\sigma') = \hat{\delta}(\sigma,\sigma'). \quad (2.16)$$

Since $\hat{\delta}(\sigma,\sigma')$ have to be equivalent to the ordinary delta function $\delta(\sigma-\sigma')$ at bulk, we use the ansatz

$$\hat{\delta}(\sigma,\sigma') = \delta(\sigma-\sigma') + a_1\delta(\sigma+\sigma') + a_2\delta(\sigma+\sigma'-2\pi), \quad (2.17)$$

where a_1 and a_2 are constants to be determined. In the neighborhood of $\sigma = 0$,

$$\hat{\delta}(\sigma, \sigma') = \delta(\sigma - \sigma') + a_1 \delta(\sigma + \sigma'). \tag{2.18}$$

Although the delta function is not a periodic function, we calculate the Fourier expansion as if it has periodicity 2π. The Fourier expansions of the delta functions are

$$\delta(\sigma - \sigma') \tag{2.19}$$
$$= \frac{1}{\pi}\left[1 + 2\sum_{n=1}^{\infty}(\cos n\sigma \cos n\sigma' + \sin n\sigma \sin n\sigma')\right],$$
$$\delta(\sigma + \sigma') \tag{2.20}$$
$$= \frac{1}{\pi}\left[1 + 2\sum_{n=1}^{\infty}(\cos n\sigma \cos n\sigma' - \sin n\sigma \sin n\sigma')\right].$$

In order to satisfy the left equation of (2.16), $a_1 = 1$. Similarly we have $a_2 = 1$. Then we have

$$\hat{\delta}(\sigma, \sigma') = \delta(\sigma - \sigma') + \delta(\sigma + \sigma') + \delta(\sigma + \sigma' - 2\pi). \tag{2.21}$$

From this and (2.16) we obtain

$$Q(0,0) = -Q(\pi,\pi) = -2, \tag{2.22}$$

where we have assumed that $Q(0,\pi)$ and $Q(\pi,0)$ are zero.

It is easy to find that the boundary Poisson bracket and other constraints (2.7) and (2.8) are consistent. The procedure will be applicable for other systems with boundary conditions which are linear in the canonical variables.

3 Membranes and constant 3-form fields

As a next step we consider a boundary Poisson bracket of an open membrane. Since membranes are 2-dimensionally extended objects, they are coupled with 3-form fields. Our aim in the present section is to construct a boundary Poisson bracket for an open membrane with a constant 3-form C-field background. Due to the 3-form field, a boundary term appears in the action of an open membrane, which is third order in its membrane's coordinates. So its boundary condition becomes non-linear and becomes a mixed type condition. By the non-linearity, a conventional Dirac procedure is not an easy task [13,14]. It is hard to construct all of the secondary constraints. We follow the previous procedure also in the present case. In addition to this, we use canonical transformation in order to make calculations possible.

We consider a membrane whose topology is cylindrical. The worldvolume coordinates of the membrane are parameterized by τ, $\sigma_1 \in [0,\pi]$ and $\sigma_2 \in [0,2\pi]$. Along the σ_2 direction the membrane is periodic.

We use the Polyakov action [16]

$$S = -\frac{T_m}{2}\int \mathrm{d}^3\xi \left[\sqrt{-\det\gamma_{\alpha\beta}}(\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu - 1) + \frac{2}{3!}\epsilon^{\alpha\beta\gamma}C_{\mu\nu\rho}\partial_\alpha X^\mu \partial_\beta X^\nu \partial_\gamma X^\rho\right], \tag{3.1}$$

where $\mathrm{d}^3\xi \equiv \mathrm{d}\tau\mathrm{d}\sigma_1\mathrm{d}\sigma_2$, T_m is the membrane tension and $\gamma_{\alpha\beta}$ is the metric tensor on the membrane. $C_{\mu\nu\rho}$ is a constant background field. This action has the worldvolume reparametrization invariance. Then we need to perform a gauge fixing procedure. Let us adopt the gauge condition: $\gamma_{0a} = 0$ $\gamma_{00} = -\det h_{ab}$ with $a,b = 1,2$. Here h_{ab} is the induced metric on the membrane. The gauge fixed action is

$$S = \frac{T_m}{2}\int \mathrm{d}^3\xi \left[\partial_\tau X^\mu \partial_\tau X_\mu - \frac{1}{2}\{X^\mu, X^\nu\}\{X_\mu, X_\nu\} - \frac{2}{3!}\epsilon^{\alpha\beta\gamma}C_{\mu\nu\rho}\partial_\alpha X^\mu \partial_\beta X^\nu \partial_\gamma X^\rho\right], \tag{3.2}$$

where $\{f,g\} = \epsilon^{ab}\partial_a f \partial_b g$.

The canonical Poisson bracket for the cylindrical membrane is changed by the boundary condition, and also by the fact that the membrane is periodic along the σ_2 direction. Before studying the boundary condition at $\sigma_1 = 0,\pi$, we see the modification of the canonical Poisson bracket due to the periodicity along the σ_2 direction.

The left hand side of the canonical Poisson bracket

$$\{X^\mu(\xi), P_\nu(\xi')\}_p = \delta^\mu_\nu \delta(\xi - \xi') \tag{3.3}$$

has periodicity 2π along σ_2 and σ_2'. However, the right hand side does not have such a periodicity. So the delta function $\delta(\sigma_2 - \sigma_2')$ has to be replaced by a periodic one, $\tilde{\delta}(\sigma_2 - \sigma_2')$, which satisfies

$$\tilde{\delta}(\sigma_2 - \sigma_2') = \tilde{\delta}(\sigma_2 - \sigma_2' + 2\pi m) \quad (m \in \mathbf{Z}). \tag{3.4}$$

Note that the canonical Poisson bracket for the cylindrical membrane is not changed only at the boundaries of the membrane but also at the bulk of it.

Next we would like to see a change of the canonical Poisson bracket by a boundary condition. For the action (3.2), the canonical Hamiltonian is

$$\mathcal{H} = \frac{T_m}{2}\left(\frac{1}{T_m}\Pi_\mu + \frac{1}{2}C_{\mu\nu\rho}\{X^\nu, X^\rho\}\right)^2 + \frac{T_m}{4}\{X^\mu, X^\nu\}^2. \tag{3.5}$$

Notice that the Hamilton theory (3.5) is canonically equivalent to that with $C_{\mu\nu\rho} = 0$. They are related by the transformation

$$\tilde{\Pi}_\mu = \Pi_\mu + \frac{1}{2}C_{\mu\nu\rho}\{X^\nu, X^\rho\}, \quad \tilde{X}^\mu = X^\mu. \tag{3.6}$$

We will start from the system $(\tilde{X}, \tilde{\Pi})$ with $C_{\mu\nu\rho} = 0$.

With respect to the membrane, we adopt the boundary condition $\partial_{\sigma_1}\tilde{X}^\mu|_{\sigma_1=0,\pi} = 0$. The secondary constraint is $\partial_{\sigma_1}\tilde{\Pi}^\mu|_{\sigma_1=0,\pi} = 0$. Furthermore the phase space (X,Π) can be transformed by $X^\mu = \hat{\Pi}^\mu$ and $\Pi^\mu = -\hat{X}^\mu$ which are canonical ones. In terms of the canonical variables $(\hat{X}, \hat{\Pi})$, the conditions are replaced by

$$\partial_1 \hat{\Pi}^\mu\Big|_{\sigma_1=0,\pi} = 0, \tag{3.7}$$

$$\partial_1 \hat{X}_\mu - C_{\mu\nu\rho}\partial_1^2 \hat{\Pi}^\nu \partial_2 \hat{\Pi}^\rho\Big|_{\sigma_1=0,\pi} = 0. \tag{3.8}$$

In order to determine the boundary Poisson bracket of the membrane, we follow the method used in the previous section with the conditions (3.7) and (3.8). The result is

$$\{\hat{\Pi}_\mu(\xi), \hat{\Pi}_\nu(\xi')\}_b = 0, \tag{3.9}$$

$$\{\hat{X}^\mu(\xi), \hat{\Pi}_\nu(\xi')\}_b = \delta^\mu_\nu \hat{\delta}(\xi - \xi'), \tag{3.10}$$

$$\begin{aligned}\{\partial_1 \hat{X}_\mu(\xi), \hat{X}_\nu(\xi')\}_\mathrm{b} = &-T_m C_{\mu\nu\rho}[\partial_1^2 \hat{\delta}(\xi - \xi')\partial_2 \hat{\Pi}^\rho(\xi) \\ &- \partial_1^2 \hat{\Pi}^\rho \partial_2 \hat{\delta}(\xi - \xi')],\end{aligned} \tag{3.11}$$

with $\hat{\delta}(\xi - \xi') := \hat{\delta}(\sigma_1, \sigma_1')\tilde{\delta}(\sigma_2 - \sigma_2')$. The last one is valid only at $\sigma_1 = 0, \pi$. We have obtained the boundary Poisson bracket of the membrane without using any approximations. When the C-field has a non-zero value, the boundaries of the membrane become non-commutative.

4 Summary

In this paper, we have explicitly shown that though there exist infinitely many secondary constraints, the boundary Poisson bracket of a bosonic open string can be determined only from two constraints by demanding its locality. The brackets (2.13)–(2.15) coincide with all other secondary constraints (2.7) and (2.8). In other words, the boundary Poisson bracket between canonical variables and the secondary constraints vanishes.

Secondly we have applied the procedure to the system of a membrane with a constant 3-form field background by using the canonical transformations. The non-commutativity depends on the canonical momentum, and it is first order in $C_{\mu\nu\rho}$.

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